REFERENCES

1. ZAIDEL R. M. and ZEL'DOVICH Ya. B., On possible regimes of steady combustion. *Prikl. Mat. Teor. Fiz.*, No. 4, 27–32, 1962.
2. KHAIKIN B. I. and RUMANOV E. N., An exothermic reaction in one-dimensional flow. *Fiz. Gor. Vzryva* **11**, 671–677, 1975.
3. VOL'PERT V. I. and VOL'PERT A. I., The existence and stability of travelling waves in chemical kinetics. In *Dynamics of Chemical and Biological Systems*. Nauka, Novosibirsk, 1989.
4. VOL'PERT V. I. and VOL'PERT A. I., Existence and stability of waves in chemical kinetics. *Khim. Fiz.* **9**, 238–245, 1990.
5. VOL'PERT V. I. and VOL'PERT A. I., Chemical conversion waves with complex kinetics. *Dokl. Akad. Nauk SSSR* **309**, 125–128, 1989.
6. VOL'PERT V. I. and VOL'PERT A. I., Some mathematical problems on wave propagation in chemically active media. *Khim. Fiz.* **9**, 1118–1127, 1990.
7. REDHEFFER R. and WALTER W., Invariant sets for systems of partial differential equations. I: Parabolic equations. *Arch. Ration. Mech. Anal.* **67**, 41–52, 1977.

Translated by R.L.Z.

J. Appl. Maths Mechs Vol. 56, No. 3, pp. 460–463, 1992
Printed in Great Britain.
0021–8928/92 $15.00 + 0.00

STRESSES ON THE SURFACE OF A RIGID NEEDLE IN AN ORTHOTROPIC ELASTIC MEDIUM†

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(*Received* 14 *May* 1990)

Using the general solution of the problem of stress concentrations on the surfaces of rigid ellipsoidal inclusions [1], the three-dimensional problem of stresses on the surface of a completely rigid needle in an unbounded elastic orthotropic medium under the action of a uniform external field is solved. By a needle we mean an ellipsoidal inclusion, one dimension of which is large compared with the other two. Explicit formulas are obtained and investigated for stresses along the principal sections of the needle in the orthotropic medium and over the entire surface of the needle in an isotropic medium. The calculations are performed, apart from the singular terms (large, but finite quantities).

1. The stress $\sigma^{\alpha\beta}(\mathbf{n})$ on the surface of a completely rigid ellipsoidal inhomogeneity in an arbitrary anisotropic medium and a uniform external field $\sigma_0^{\alpha\beta}$ has the form

$$\sigma(\mathbf{n})=F(\mathbf{n})\sigma_0,\quad F(\mathbf{n})=D(\mathbf{n})R,\quad D(\mathbf{n})=cK(\mathbf{n}) \tag{1.1}$$

Here $\mathbf{n}=(n_1, n_2, n_3)$ is the limit normal vector to the ellipsoidal surface with semi-axes a_α ($\alpha = 1, 2, 3$) and $F(\mathbf{n})$ is a tensor concentration coefficient. The tensor $K(\mathbf{n})$ does not depend on the geometry of the inhomogeneity, is expressed in terms of the Fourier transform of the Green tensor of the homogeneous medium, and was obtained explicitly in [1] for an orthotropic medium. The tensor of elastic constants c of the

† *Prikl. Mat. Mekh.* Vol. 56, No. 3, pp. 549–552, 1992.

orthotropic medium in a system of coordinates coupled rigidly to the axes of the ellipsoid has nine non-zero components which we denote by

$$c^{\alpha\alpha\beta\beta}=c_{\alpha\beta}(\alpha,\ \beta=1,\ 2,\ 3),\ c^{2323}=c_{44},\ c^{1313}=c_{55},\ c^{1212}=c_{66}$$

The tensor $D(\mathbf{n})$ is found by contracting the tensors c and $K(\mathbf{n})$.

The most difficult part of finding the concentration coefficients is the calculation of the tensor R in (1.1). The four-tensor R depends explicitly on the geometrical parameters of the ellipsoid and is the inverse tensor to $D=\langle D(\mathbf{n})\rangle$, where $\langle D(\mathbf{n})\rangle$ is the mean value of $D(\mathbf{n})$ over the surface of the ellipsoid [2, 3]. Just like $D(\mathbf{n})$, D and R are symmetrical within pairs of indices, but index pairs cannot be transposed.

We introduce the dimensionless parameters

$$\eta=a_2a_1^{-1},\quad \xi=a_3a_2^{-1}\quad (a_1\geqslant a_2\geqslant a_3)$$

For needle $\eta\ll 1$, $\xi\sim 1$, and the expansion of the tensor D in terms of the small (but finite) parameter η has the form

$$D=D_0+\eta^2|\ln\eta|D_1+O(\eta^2)$$

$$D_0=\xi\pi^{-1}\int_0^\pi D(\varphi,0)(\cos^2\varphi+\xi^2\sin^2\varphi)^{-1}d\varphi \tag{1.2}$$

$$D_1=\xi\pi^{-1}\int_0^\pi D_{tt}''(\varphi,0)\,d\varphi \tag{1.3}$$

$$D(\varphi,0)=D(n_1\equiv t=0,\ n_2=\cos\varphi,\ n_3=\sin\varphi)$$

It has been shown [1] that the tensor D_0 does not have an inverse and singularities of order $(\eta^2\ln\eta)^{-1}$ appear in the components $R^{11}{}_{\cdot\cdot\alpha\alpha}$. Corresponding singularities appear in the stresses in neighbourhoods of the needle endfaces from the extension $\sigma_0{}^{\alpha\alpha}$ ($\alpha=1,\ 2,\ 3$). The pure shear components of the external stresses do not produce singularities on the surface of the needle.

Assuming that the external medium is orthotropic, we will calculate the components $R^{11}{}_{\cdot\cdot\alpha\alpha}$, apart from the singular terms.

Analysis of the structure of the tensor D shows that $(D_0)^{\alpha\alpha}{}_{\cdot\cdot 11}=0$, $(D_0)^{\alpha\alpha}{}_{\cdot\cdot 22}\neq 0$ and $(D_0)^{\alpha\alpha}{}_{\cdot\cdot 33}\neq 0$ and to determine the singular terms of $R^{11}{}_{\cdot\cdot\alpha\alpha}$ it is sufficient to find $(D_0)^{\alpha\alpha}{}_{\cdot\cdot 22}$, $(D_0)^{\alpha\alpha}{}_{\cdot\cdot 33}$ and $(D_1)^{\alpha\alpha}{}_{\cdot\cdot 11}$ expressed in terms of the single integrals (1.2) and (1.3).

We will calculate the tensor D_0. For later convenience we will derive $D_0\xi^{-1}$ rather than D_0:

$$D_0\xi^{-1}=L^{-1}(\sqrt{c_{33}}+L\xi+\xi^2\sqrt{c_{22}})^{-1}[Q+B\sqrt{c_{33}}(\xi^{-1}L+\sqrt{c_{22}})+$$

$$+N\sqrt{c}\,(L\xi+\sqrt{c_{33}})] \tag{1.4}$$

$$L=[\Delta_{11}c+2((c_{22}c_{33})^{1/2}-c_{23})]^{1/2}$$

The non-zero components of the tensors B, Q and N have the form

$$Q^{11}{}_{\cdot\cdot 22}=-(\Delta_{12}c_{44}{}^{-1}+c_{13}),\quad B^{22}{}_{\cdot\cdot 33}=N^{33}{}_{\cdot\cdot 22}=c_{23},\ N^{11}{}_{\cdot\cdot 22}=c_{12}$$

$$Q^{11}{}_{\cdot\cdot 33}=-(\Delta_{13}c_{44}{}^{-1}+c_{12}),\quad B^{33}{}_{\cdot\cdot 33}=-Q^{33}{}_{\cdot\cdot 22}=c_{33},\quad B^{11}{}_{\cdot\cdot 33}=c_{13}$$

$$Q^{22}{}_{\cdot\cdot 22}=Q^{33}{}_{\cdot\cdot 33}=\Delta_{11}c_{44}{}^{-1}-c_{23}\equiv k_1,\quad N^{22}{}_{\cdot\cdot 22}=-Q^{22}{}_{\cdot\cdot 33}=c_{22}$$

$\Delta_{\alpha\beta}$ is the cofactor of the element $c_{\alpha\beta}$ in the matrix $\|c_{\alpha\beta}\|$.

We will find $(D_1)^{\alpha\alpha}{}_{\cdot\cdot 11}$. Investigation of the structure of the tensor $D_{tt}''(\phi,0)$ from (1.3) has shown that the ϕ-dependence in integrals (1.2) and (1.3) is analogous and the components of $(D_1\xi^{-1})$ are obtained from $(D_0\xi^{-1})$ by multiplying the right-hand side of (1.4) by $2c_{66}{}^{-1}$ and replacing ξ^2 in it by $c_{55}c_{66}{}^{-1}$, and the tensors B, Q, N by T, P, S, respectively. The non-zero components of T, P, S are as follows:

$$T^{11}{}_{\cdot\cdot 11}=\Delta_{33}-c_{12}c_{66},\quad T^{22}{}_{\cdot\cdot 11}=-c_{22}c_{66}$$

$$T^{33}{}_{\cdot\cdot 11}=-(\Delta_{13}+c_{23}c_{66}),\ P^{33}{}_{\cdot\cdot 11}=-c_{33}c_{55}$$

$$S^{11}{}_{\cdot\cdot 11}=c_{44}{}^{-1}(\Delta+c_{55}\Delta_{13}+c_{66}\Delta_{12})+c_{12}c_{55}+c_{13}c_{66}+2\Delta_{23}$$

$$S^{22}{}_{\cdot\cdot 11}=c_{22}c_{55}-\Delta_{13}-k_1c_{66},\quad P^{11}{}_{\cdot\cdot 11}=k_2c_{55}$$

$$S^{33}{}_{\cdot\cdot 11}=c_{33}c_{66}-\Delta_{12}-k_1c_{55},\quad P^{22}{}_{\cdot\cdot 11}=k_3c_{55}$$

$$k_2=\Delta_{22}c_{55}{}^{-1}-c_{13},\quad k_3=-(\Delta_{12}c_{55}{}^{-1}+c_{23})$$

The components $R^{11}{}_{..\alpha\alpha}$ are elements of the inverse matrix to $\|b_{\alpha\beta}\|$, $(\alpha, \beta = 1, 2, 3)$, where

$$b_{\alpha 1}=\eta^2|\ln \eta|\ (D_1)^{\alpha\alpha}{}_{..11},\ b_{\alpha 2}=(D_0)^{\alpha\alpha}{}_{..22},\ b_{\alpha 3}=(D_0)^{\alpha\alpha}{}_{..33}$$

Inverting $\|b_{\alpha\beta}\|$ we obtain

$$R^{11}{}_{..\alpha\alpha}=\Delta^{-1}\Delta_{1\alpha}\varkappa\sqrt{c_{55}c_{66}},\quad \varkappa=(\xi\eta^2|\ln \eta|)^{-1}$$

where Δ is the determinant of the matrix $\|c_{\alpha\beta}\|$ $(\alpha, \beta = 1, 2, 3)$.

If we introduce Young's moduli E_α, the Poisson's ratios $\nu_{\alpha\beta}$ and the shear moduli $G_{\alpha\beta}$ of an orthotropic medium [4], then the components $R^{11}{}_{..\alpha\alpha}$ acquire the form

$$R^{11}{}_{..11}=\varkappa\sqrt{G_{12}G_{13}}E_1{}^{-1},\quad R^{11}{}_{..\gamma\gamma}=-\nu_{1\gamma}R^{11}{}_{..11}\quad (\gamma=2,\ 3)$$

For an isotropic medium with Poisson's ratio ν

$$R^{11}{}_{..22}=R^{11}{}_{..33}=-\nu R^{11}{}_{..11}=-0{,}5\ \nu\varkappa(1+\nu)^{-1}$$

2. The singular components of the stress on the surface of the needle have the form

$$\sigma^{\alpha\beta}(\mathbf{n})=D^{\alpha\beta}{}_{..11}(\mathbf{n})R^{11}{}_{..\lambda\lambda}\sigma_0{}^{\lambda\lambda}$$

We will derive the values of the stresses at the endface $A(a_1, 0, 0)$ of the needle in the orthotropic medium:

$$\begin{aligned}
\sigma^{11}(A)&=\Delta^{-1}\varkappa\sqrt{c_{55}c_{66}}(\Delta_{11}\sigma_0{}^{11}+\Delta_{12}\sigma_0{}^{22}+\Delta_{13}\sigma_0{}^{33})=\\
&=E_1{}^{-1}\varkappa\sqrt{G_{12}G_{13}}(\sigma_0{}^{11}-\nu_{12}\sigma_0{}^{22}-\nu_{13}\sigma_0{}^{33})\\
\sigma^{22}(A)&=c_{12}c_{11}{}^{-1}\sigma^{11}(A)=\\
&=(\nu_{21}+\nu_{23}\nu_{31})(1-\nu_{23}\nu_{32})^{-1}\sigma^{11}(A)\\
\sigma^{33}(A)&=c_{13}c_{11}{}^{-1}\sigma^{11}(A)=\\
&=(\nu_{31}+\nu_{32}\nu_{21})(1-\nu_{23}\nu_{32})^{-1}\sigma^{11}(A),\quad \sigma^{\alpha\beta}(A)=0\quad (\alpha\neq\beta)
\end{aligned}\tag{2.1}$$

For an isotropic medium

$$\begin{aligned}
&\sigma^{11}(A)={}^1\!/_2\varkappa(1+\nu)^{-1}[\sigma_0{}^{11}-\nu(\sigma_0{}^{22}+\sigma_0{}^{33})]\\
&\sigma^{22}(A)=\sigma^{33}(A)=\nu(1-\nu)^{-1}\sigma^{11}(A),\quad \sigma^{\alpha\beta}(A)=0\quad (\alpha\neq\beta)
\end{aligned}\tag{2.2}$$

Investigation of the dependence of the needle stresses on the form of the external field and medium anisotropy will be performed in a local system of coordinates fixed to the surface normal $\mathbf{n}$ so that the e_1 axis is directed along $\mathbf{n}$, while the e_2 and e_3 axes lie in the tangent plane. We denote the stresses in the local system by $\sigma_{\alpha\beta}(\mathbf{n})$. Then $\sigma_{11}(\mathbf{n})$ is directed along the normal to the surface at all its points. In the $n_2 = 0$ section the stress $\sigma_{22}(\mathbf{n})$ is directed orthogonally to the plane of the section and $\sigma_{33}(\mathbf{n})$ along the contour of the section; and conversely for $n_3 = 0$. On the needle endface $\sigma_{\alpha\beta}(A) = \sigma^{\alpha\beta}(A)$.

For an isotropic medium we find that over the whole surface of the needle

$$\begin{aligned}
&\sigma_{11}(\mathbf{n})=n_1{}^2\sigma^{11}(A),\quad \sigma_{23}(\mathbf{n})=0\\
&\sigma_{22}(\mathbf{n})=\sigma_{33}(\mathbf{n})=\nu(1-\nu)^{-1}n_1{}^2\sigma^{11}(A)\\
&\sigma_{12}(\mathbf{n})=n_1n_2\sqrt{((1-n_3{}^2)^{-1}}\sigma^{11}(A)\\
&\sigma_{13}(\mathbf{n})=n_1{}^2n_3\sqrt{(1-n_3{}^2)^{-1}}\sigma^{11}(A)
\end{aligned}\tag{2.3}$$

with the expression for $\sigma^{11}(A)$ being given by (2.2).

For an orthotropic medium we derive formulas for the stresses along the principal sections of the needle. In the central section $n_1 = 0$ the singular stresses vanish, while for $n_2 = 0$ we have

$$\begin{aligned}
&\sigma_{11}(\mathbf{n})=n_1{}^2\sigma^{11}(A)\\
&\sigma_{22}(\mathbf{n})=pn_1{}^2(c_{12}n_1{}^2+k_3n_3{}^2)\sigma^{11}(A)\\
&\sigma_{33}(\mathbf{n})=pn_1{}^2[c_{13}n_1{}^4+(k_2-2c_{33})n_3{}^4+\\
&\quad+(c_{11}-c_{33}+2c_{13})n_1{}^2n_3{}^2]\sigma^{11}(A)\\
&\sigma_{13}(\mathbf{n})=n_1n_3\sigma^{11}(A),\quad \sigma_{12}(\mathbf{n})=\sigma_{23}(\mathbf{n})=0\\
&p=[c_{11}n_1{}^4+c_{33}n_3{}^4+(k_2-c_{13})n_1{}^2n_3{}^2]^{-1}
\end{aligned}\tag{2.4}$$

where the expression for $\sigma^{11}(A)$ was given by the first formula in (2.1).

In the $n_3 = 0$ section the expression for $\sigma_{\alpha\beta}(\mathbf{n})$ is obtained by exchanging indices $2 \leftrightarrow 3$ and $5 \leftrightarrow 6$ on both sides of Eqs (2.4).

It is clear from the formulas derived that all the singular surface stresses are expressed in terms of the stress $\sigma^{11}(A)$ at the endface of the needle. Hence some qualitative results can be obtained simply by investigating $\sigma^{11}(A)$.

The stress $\sigma^{11}(A)$ is directed along the major axis, is (in absolute terms) the largest stress at the endface of the needle, and can have either the same sign as the external field or the opposite sign for small σ_0^{11} and sufficiently large σ_0^{22}, σ_0^{33}. The value of $\sigma^{11}(A)$ depends only on the elastic constants describing the properties of the medium and the direction of the axis of the needle, with $\sigma^{11}(A)$ increasing as Young's modulus E_1 decreases and as the shear moduli G_{12}, and G_{13} increase.

In the transition from a needle $\xi \leqslant 1$ to a stretched disk $(\xi \ll 1)$ the stress over the entire surface increases as ξ^{-1}.

Investigation of the dependence of the stresses on the surface normal shows that the qualitative picture of the distribution of the stresses over the needle in an orthotropic medium can be distinguished from the isotropic case. For an isotropic medium the stresses $\sigma_{\alpha\alpha}(\mathbf{n})$ $(\alpha = 1, 2, 3)$ reach their highest values at the endface, while for $\sigma_{12}(\mathbf{n})$ and $\sigma_{13}(\mathbf{n})$ there is a "splash" effect [3]. For an orthotropic medium the behaviour of the stress $\sigma_{11}(\mathbf{n})$ normal to the surface is similar to the isotropic case, while for the stresses $\sigma_{22}(\mathbf{n})$ and $\sigma_{33}(\mathbf{n})$, as for a hollow needle [5], the maximum can be shifted from the end of the needle.

REFERENCES

1. MIRENKOVA G. N. and SOSNINA E. G., A rigid ellipsoidal disk and needle in an anisotropic elastic medium. *Prikl. Mat. Mekh.* **45**, 165–170, 1981.
2. KUNIN I. A. and SOSNINA E. G., Stress concentration on an ellipsoidal inhomogeneity in an anisotropic elastic medium. *Prikl. Mat. Mekh.* **37**, 306–315, 1973.
3. KUNIN I. A., MIRENKOVA G. N. and SOSNINA E. G., An ellipsoidal crack and needle in an anisotropic elastic medium. *Prikl. Mat. Mekh.* **37**, 524–531, 1973.
4. LEKHNITSKII S. G., *Elasticity Theory for an Anisotropic Body*. Nauka, Moscow, 1977.
5. MIRENKOVA G. N. and SOSNINA E. G., A hollow ellipsoidal needle in an orthotropic elastic medium. *Prikl. Mat. Mekh.* **54**, 1017–1021, 1990.

Translated by R.L.Z.